Randomisation de la NTT pour les cryptosystèmes basés sur RLWE pour contrer les attaques par canaux cachés basées sur la Belief Propagation

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Outline

1. RLWE cryptosystem and NTT
2. Belief propagation on NTT
3. Counter-measures
4. Simulation results and conclusion

Outline

Notations and RLWE problem

Notations

- p a prime integer.
- $n = 2^t$.
- $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$.
- $\mathcal{R}_p = \mathbb{Z}_p[X]/(X^n - 1)$.
- $A \overset{\mathcal{U}}{\leftarrow} \mathcal{R}_p$ uniform distribution.
- $E \overset{\chi}{\leftarrow} \mathcal{R}_p$ distribution with small coefficients.

RLWE problem.

$A \overset{\mathcal{U}}{\leftarrow} \mathcal{R}_p$ and $S, E \overset{\chi}{\leftarrow} \mathcal{R}_p$ find S from:

$$A \text{ and } B = A \times S + E$$

RLWE cryptosystem

Alice → Bob

$$Pk = (A, B)$$

Key Generation

$$A \xleftarrow{\mathcal{U}} \mathcal{R}_p \text{ and } S, E \xleftarrow{\chi} \mathcal{R}_p$$

$$B = A \times S + E$$

Encryption

Plaintext $\mathrm{m} \in \mathcal{R}_p$

$$C_1 = A \times E_1 + E_2,$$

$$C_2 = B \times E_1 + E_3 + m$$

Decryption

$$\mathrm{m} = Round(C_2 - C_1 \times \mathbf{S}) \longleftarrow (C_1, C_2)$$

Multiplication in $\mathcal{R}_p$ with NTT/FFT

We recall $\mathcal{R}_p = \mathbb{Z}_p[X]/(X^n - 1)$ and ω is a primitive n-th root of unity. We have

$$(X^n - 1) = (X - \omega^0)(X - \omega^1)\cdots(X - \omega^{n-1})$$

$$\mathcal{R}_p \cong \mathbb{Z}_p[X]/(X - \omega^0) \times \mathbb{Z}_p[X]/(X - \omega^1) \times \cdots \times \mathbb{Z}_p[X]/(X - \omega^{n-1})$$

- The multiplication of F and G in $\mathcal{R}_p$ can be done as follows

$$\begin{array}{rcl}
F(X) & \xrightarrow{evaluation} & \hat{F} = (F(\omega^0), F(\omega^1), \ldots, F(\omega^{n-1})) \\
G(X) & \xrightarrow{evaluation} & \hat{G} = (G(\omega^0), G(\omega^1), \ldots, G(\omega^{n-1})) \\
 & & \downarrow \\
R(X) = F(X) \times G(X) & \xleftarrow{interpolation} & (\hat{f}_0 \times \hat{g}_0, \hat{f}_1 \times \hat{g}_1, \ldots, \hat{f}_{n-1} \times \hat{g}_{n-1})
\end{array}$$

FFT/NTT - Odd-even splitting

$$F(X) = \underbrace{\sum_{k=0}^{n/2} f_{2k} X^{2k}}_{F_e(X^2)} + X \times \underbrace{\sum_{k=0}^{n/2} f_{2k+1} X^{2k}}_{F_o(X^2)}$$

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For $j = 0, \ldots, n/2 - 1$

$$\begin{aligned} F(\omega^j) &= F_e(\omega^{2j}) + \omega^j F_o(\omega^{2j}) \\ F(\omega^{n/2+j}) &= F_e(\omega^{2j}) - \omega^j F_o(\omega^{2j}) \end{aligned}$$

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NTT(F, ω, n)

if $n = 1$ **then**
  return $\widehat{F} = [f_0]$
  $F_e = \sum_{k=0}^{\frac{n}{2}-1} f_{2k} X^k$
  $F_o = \sum_{k=0}^{\frac{n}{2}-1} f_{2k+1} X^k$

Recursion

$\widehat{R} \longleftarrow NTT(F_e, \omega', n/2)$
$\widehat{R'} \longleftarrow NTT(F_o, \omega', n/2)$

for $i = 0; i < n/2;$i++ **do**
  $\widehat{F}_i = \widehat{r}_i + \omega^i \widehat{r}_i'$
  $\widehat{F}_{i+n/2} = \widehat{r}_i - \omega^i \widehat{r}_i'$

return $\widehat{F}$

FFT/NTT - High-Low splitting

- $F(\omega^{2j}) = \sum_{k=0}^{\frac{n}{2}-1} (f_k + f_{n/2+k})(\omega^{2j})^k$
- $F(\omega^{2j+1}) = \sum_{k=0}^{\frac{n}{2}-1} (f_k - f_{n/2+k})\omega^k(\omega^{2j})^k$

NTT(F, ω, n)

if $n = 1$ **then**
  return $\widehat{F} = [f_0]$
  $\omega' = \omega^2$
  $R = \sum_{k=0}^{\frac{n}{2}-1} (f_k + f_{n/2+k})X^k$
  $R' = \sum_{k=0}^{\frac{n}{2}-1} (f_k - f_{n/2+k})\omega^k X^k$

  Recursion

  $\widehat{R} \longleftarrow NTT(R, \omega', n/2)$
  $\widehat{R'} \longleftarrow NTT(R', \omega', n/2)$

return $\widehat{F} = (\widehat{r}_0, \widehat{r}'_0, \cdots, \widehat{r}_{n/2-1}, \widehat{r}'_{n/2-1})$

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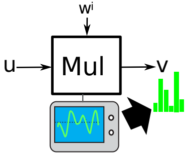
Belief propagation : leaking model and propagation

Leackage : Power consumption provides probabilities on processed values:



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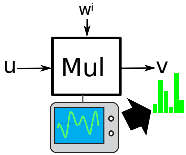
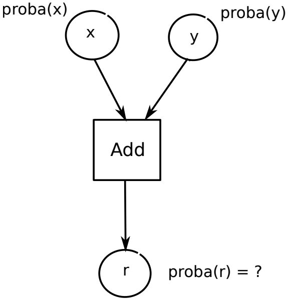


Belief propagation propagates probabilities as follows:



$$proba(r) = \sum_{x+y=r} proba(x) prob(y)$$

Belief propagation : leaking model and propagation

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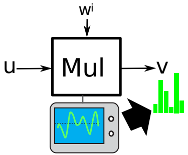

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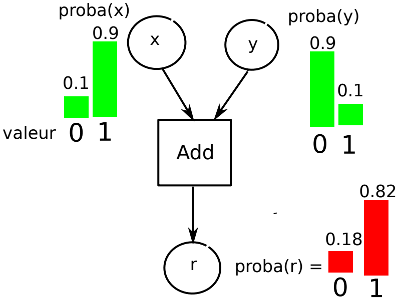

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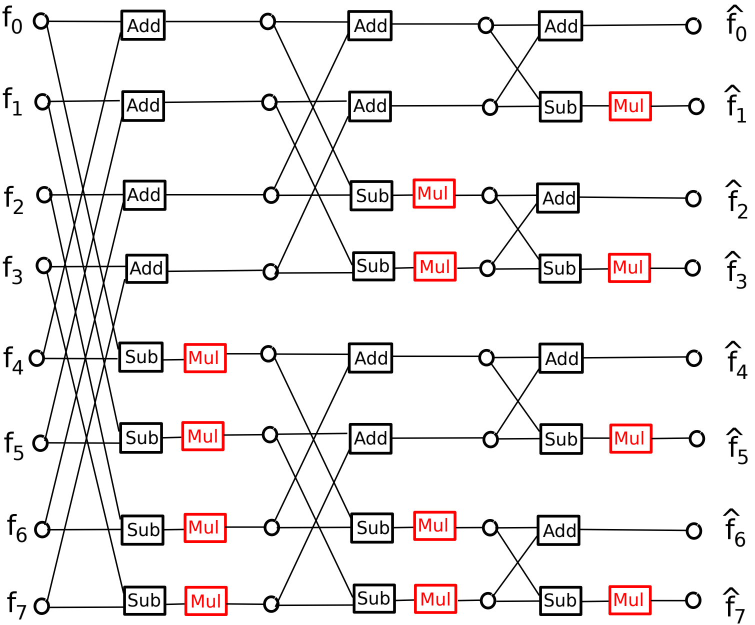
Belief-propagation on NTT



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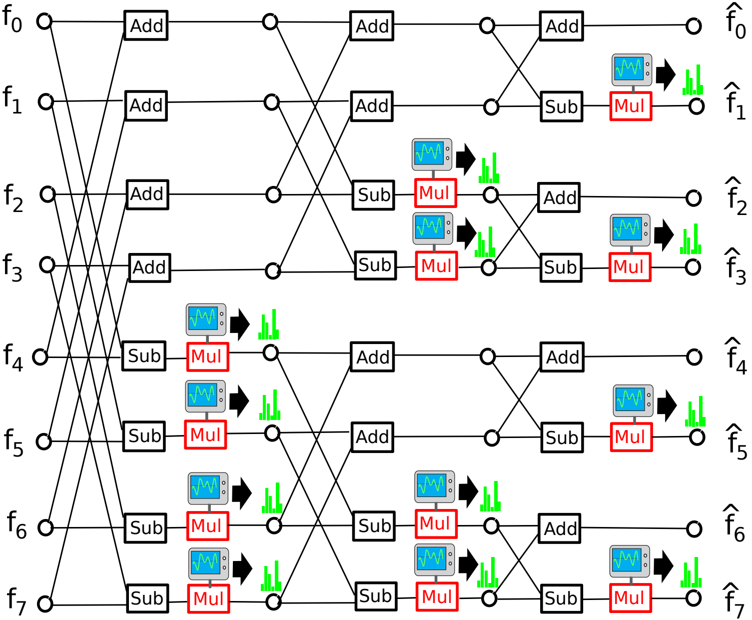

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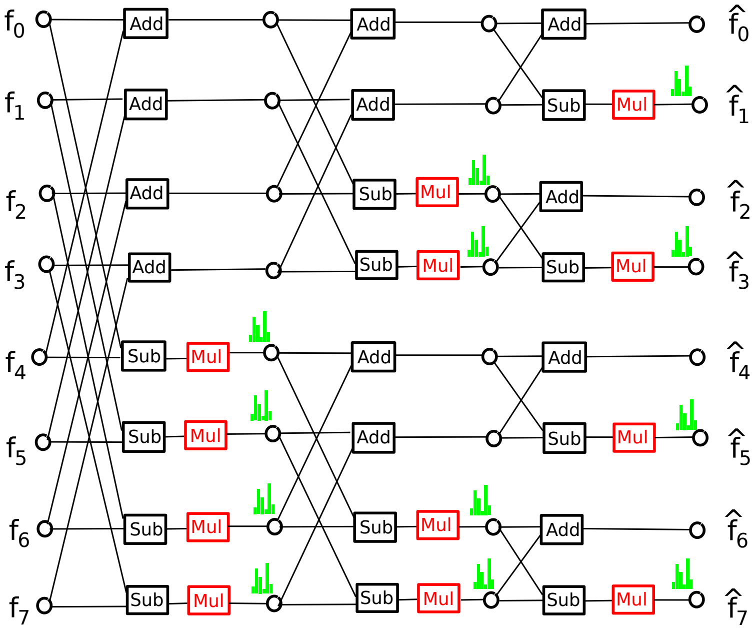

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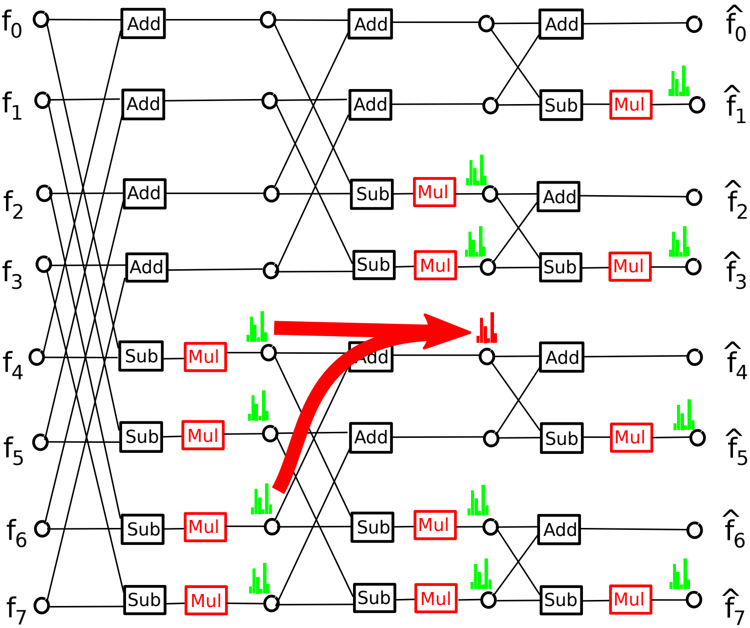

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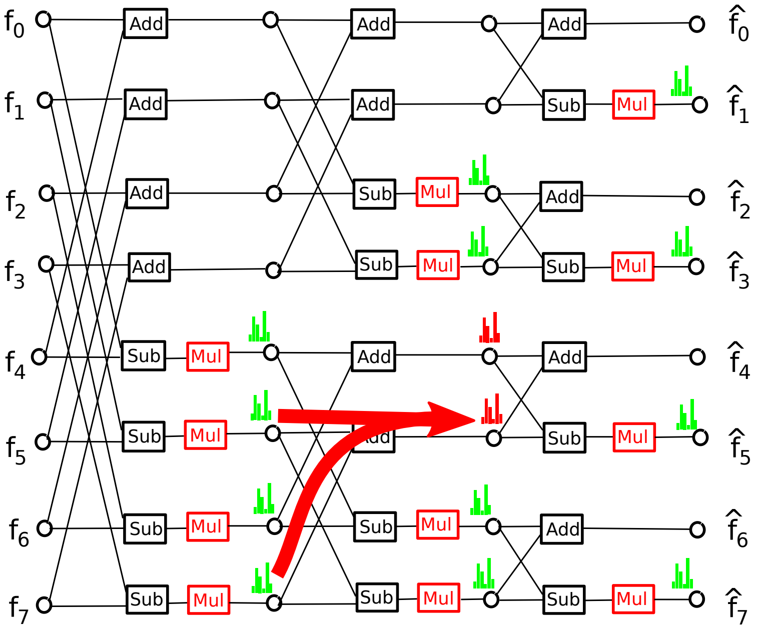

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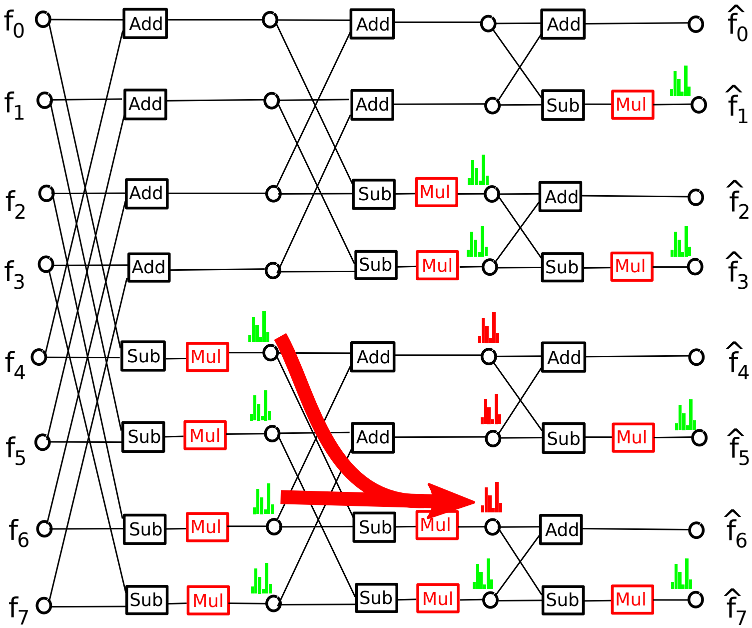

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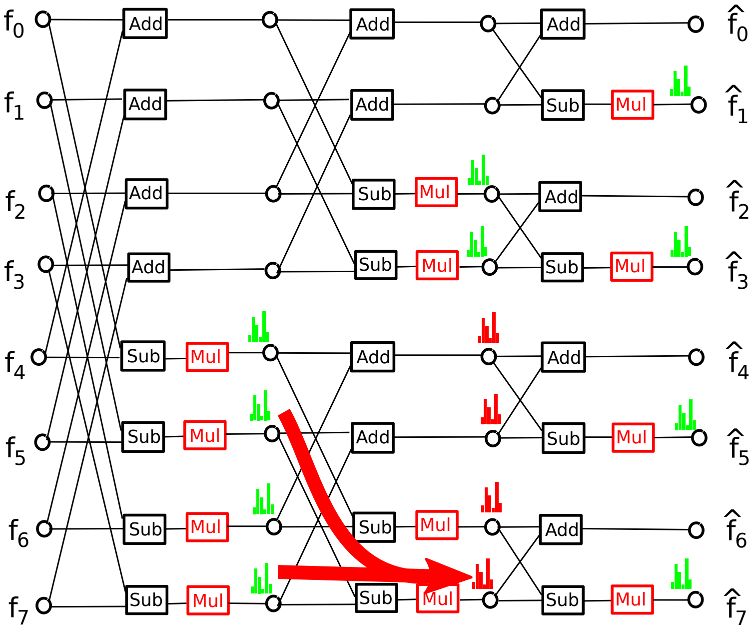

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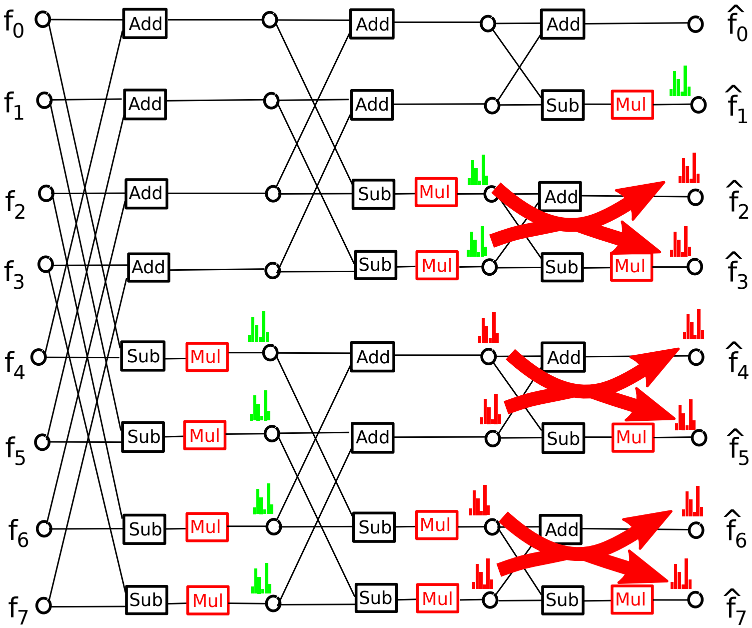

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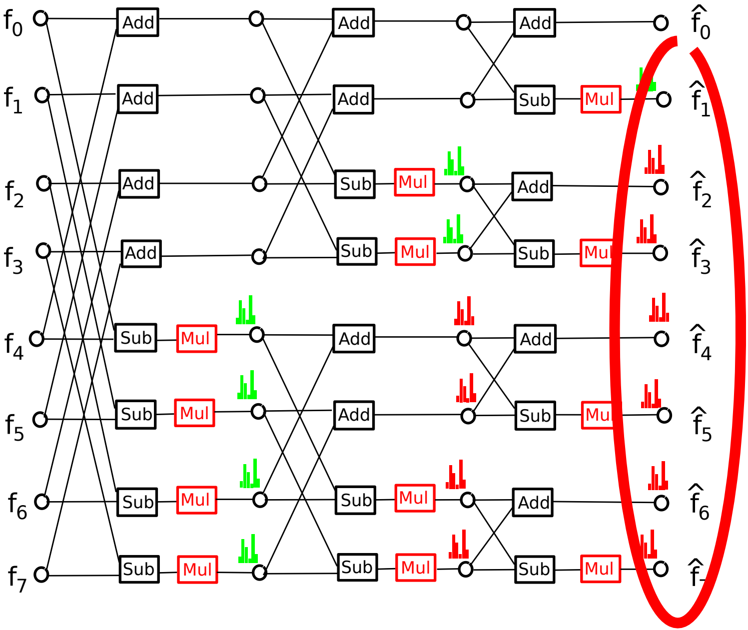


Outline

NTT randomisation : State of the art

Multiplicative mask	$\mathbf{f}' = \alpha \times \mathbf{f}$ with a random $\alpha \in \mathbb{Z}_q$ $\widehat{\mathbf{f}'} = NTT(\mathbf{f}')$ $\widehat{\mathbf{f}} = \alpha^{-1} \times \widehat{\mathbf{f}'}$
Additive mask	random split $\mathbf{f} = \mathbf{f}' + \mathbf{f}''$ $\widehat{\mathbf{f}'} = NTT(\mathbf{f}')$ $\widehat{\mathbf{f}''} = NTT(\mathbf{f}'')$ $\widehat{\mathbf{f}} = \widehat{\mathbf{f}'} + \widehat{\mathbf{f}''}$
Shifting	$\mathbf{f}' = X^r \times \mathbf{f} \mod (X^n - 1)$ $\widehat{\mathbf{f}'} = NTT(\mathbf{f}')$ $\widehat{\mathbf{f}} = (\omega^{-ir}\widehat{\mathbf{f}'}_i)_{i=0,\ldots,n-1}$
Shuffling	At each level of NTT, perform butterfly operations in random order

Proposed (virtually free) randomisation

- Randomisation by mixing HL and OE NTT.
- Randomising the root of unity ω.
- Randomising with multiplicative mask ω^i.
- Random reduction (Barrett/Montgomery).

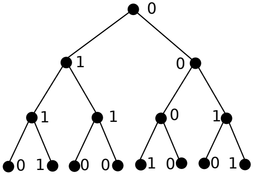
Randomization mixing High-Low and Odd-Even NTT (1/2)

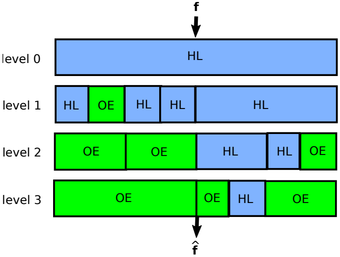
NTT-RandHLOE(F, ω, n)

if $n = 1$ **then**
  return $\widehat{F} = [f_0]$
else
  $r \xleftarrow{\mathcal{U}} \{0, 1\}$
  if $r = 0$ **then**
    // Apply High-Low splitting formula
    // and recursion
    ...
  else
    // Apply Odd-Even splitting formula
    // and recursion
    ...

Randomization mixing High-Low and Odd-Even NTT (2/2)

Random bits. (with $n = 16 = 2^4$)





Level of randomisation: $n/2$

NTT with random reduction (1/2)

p is an $\ell - 1$ bit integer.
$p' = p^{-1} \pmod{2^\ell}$

Montgomery Multiplication (MM)

1: $z \leftarrow x \times y$
2: $s \leftarrow p^{-1} \times z \pmod{2^\ell}$
3: $r \leftarrow (z - s \times p)/2^\ell$

$$r \equiv (x \times y) \times 2^{-\ell} \mod p$$

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$r \equiv (x \times y) \times 2^{-\ell} \mod p$

$p' = \lfloor 2^{2\ell}/p \rfloor$

Barrett Multiplication (BM)

1: $z \leftarrow x \times y$
2: $s \leftarrow \lfloor \lfloor z/2^{\ell-1} \rfloor p'/2^{\ell+1} \rfloor$
3: $r \leftarrow z - s \times p$

$r \equiv (x \times y) \mod p$

NTT with random reduction (2/2)

NTT-RandRed(F, ω, n)

if $n = 1$ **then**
  return $\widehat{F} = [f_0]$
else
  $r \xleftarrow{\mathcal{U}} \{0, 1\}$
  if $r = 0$ **then**
    // High-Low butterfly with BM
    // and recursion
    ...
  else
    // High-Low butterfly with MM
    // and recursion
    ...

- Level of random ($n = 2^t$) :

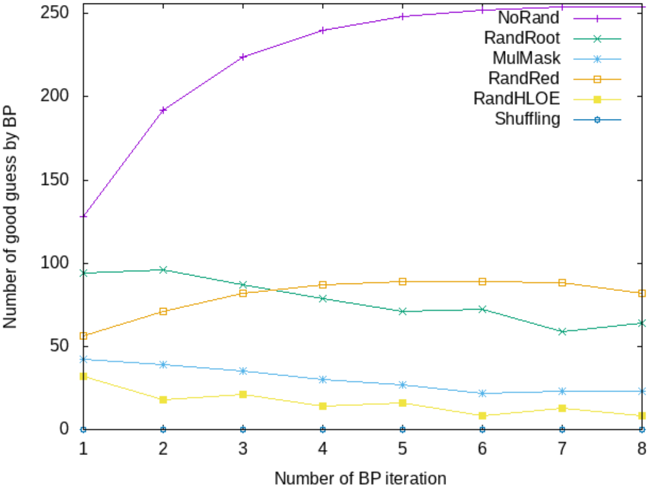
$$1+2+2^2+\ldots+2^{t-2} = \frac{n}{2} - 1$$

- Multiplicative masks are "small" :

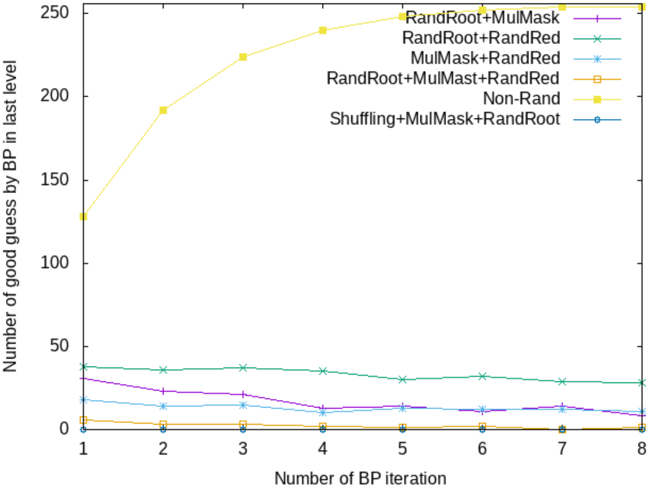
$$\hat{f}_j \times (2^{-\ell})^i, i \in \{0, \ldots, t-2\}$$

Outline

Simulation results : only the last level ($n = 256$)



Simulation results : combined randomisation ($n = 256$)



Conclusion

Level of randomization:

	Formula	$n = 256$
Shuffling	$(\frac{n}{2}!)^{\log_2(n)}$	2^{5729}
RandHLOE	$n/2$	2^7
RandRoot	$\sim \frac{2^n}{n}$	2^{247}
RandMulMask	$\sim n^{n/2}$	2^{1024}
RandRed	$n/2$	2^7

Simulation results show that:

- All tested randomisation are effective against BP.
- Combined randomisations are better.

Some remaining questions:

- Is any brute-force or trade-off attack on randomisation effective for BP ?
- The effect of randomization on the first phase of the attack: template attack on modular multiplication ?

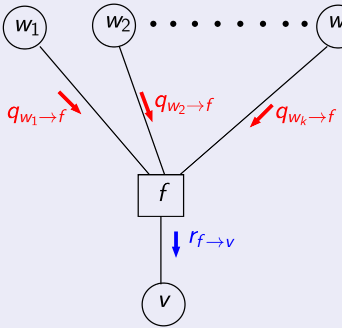
Thank you for you attention

Any question ?

Belief propagation - general setting

Graph of factor nodes □ and variable nodes ○.

Information : factor to variable



$$r_{f \to v}(x) = \sum_{x_i} f(x, x_1, .., x_k) \prod_{i=1}^{k} q_{v_i \to f}(x_i)$$

Belief propagation - general setting

Graph of factor nodes □ and variable nodes ○.

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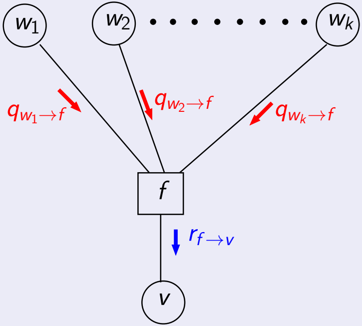
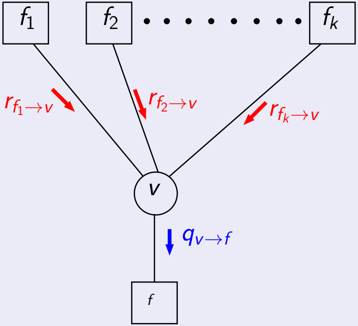


$$r_{f\to v}(x) = \sum_{x_i} f(x, x_1, .., x_k) \prod_{i=1}^{k} q_{v_i\to f}(x_i)$$

Information : variable to factor



$$q_{v\to f}(x) = \prod_{i=1}^{k} r_{f_i\to v}(x)$$

Toy example : BP on NTT for $n = 16$ and $q = 257$

- G = Good. The value with highest probability corresponds to real value computed.

Belief Probagation iteration 1 :

init	**B-B-B-B-B-B-B-B-B-B-B-B-B-B-B-B**
level 1	**B-B-B-B-B-B-B-B-G-G-G-G-G-G-G-G**
level 2	**B-B-B-B-G-G-G-G-B-B-B-B-G-G-G-G**
level 3	**B-B-G-G-B-B-G-G-B-B-G-G-B-B-G-G**
level 4	**B-B-B-B-B-B-B-B-B-B-B-B-B-B-B-B**

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level 2	B-B-B-B-G-G-G-G-B-B-B-B-G-G-G-G
level 3	B-B-G-G-B-B-G-G-B-B-G-G-B-B-G-G
level 4	B-B-B-B-B-B-B-B-B-B-B-B-B-B-B-B

After two iterations of BP:

init	B-B-B-B-B-B-B-B-B-B-B-B-B-B-B-B
level 1	B-B-B-B-B-B-B-B-G-G-G-G-G-G-G-G
level 2	B-B-B-B-G-G-G-G-G-G-G-G-G-G-G-G
level 3	B-B-G-G-G-G-G-G-B-B-G-G-G-G-G-G
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After two iterations of BP:

init	B-B-B-B-B-B-B-B-B-B-B-B-B-B-B-B
level 1	B-B-B-B-B-B-B-B-G-G-G-G-G-G-G-G
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level 4	B-B-G-G-B-B-G-G-B-B-G-G-B-B-G-G

After four iterations of BP:

init	B-B-B-B-B-B-B-B-B-B-B-B-B-B-B-B
level 1	B-B-B-B-B-B-B-B-G-G-G-G-G-G-G-G
level 2	B-B-B-B-G-G-G-G-G-G-G-G-G-G-G-G
level 3	B-B-G-G-G-G-G-G-G-G-G-G-G-G-G-G
level 4	B-B-G-G-G-G-G-G-G-G-G-G-G-G-G-G

Afterwards, there is no improvement.